Description of dental arch form using the Fourier series

The aim of this study was to describe the form of the human superior dental arch using Fourier transformations. Forty models made in dental stone from impressions of the maxillary dental arch were used to obtain the reference data, which were expressed in Cartesian coordinates, from the mesovestibular cuspid vertices of molar teeth, vestibular cuspid of premolars, and incisal edge. Fourth-grade equations and Fourier series were calculated from these data. The results indicate that Fourier series more precisely express the form and size of different dental arches, with mixed or permanent dentition, than fourth-grade equations. Details of the mathematical procedure and the precision obtained were provided. (Int J Adult Orthod Orthognath Surg 2002;17:59–65)

Mathematical descriptions of the dental arch form have been intended for more than a century. Sillman concluded after his measurements that the dental arch shape, as well as its absolute dimension, changes from the beginning of life to the time of eruption of the temporal second molars. The traditional studies of the dental arch shape, based upon the analysis of either Cartesian or lineal coordinates, are mainly centered in the description of age changes and ethnic differences. The interpretation of ethnic contrasts, although analyzed by one-way or multivared statistical models, is affected by many factors, one of them being the ample range of natural variation. The dental arch has been described as parabolic following the equation \( x^2 = -2py \); as a catenary curve of the form \( y = (e^x + e^{-x})/2 \), through mathematical equations through finite elements analysis, and by Euclidean distance matrix analysis (EDMA). Other authors have described the superior dental arch qualitatively as semi-elliptic (Black, 1894), or as “U-shaped” (Martin, 1914). Although the use of fourth-grade equations has been the method more frequently applied to express the form of the dental arch, it is desirable to find more precise methods, as the error of methods based on these equations is due to the fact that they generate curves that do not fit well to the irregularity of the real arches. The present approach intends to describe the dental arch using Fourier series.

Materials and methods

Data were gathered from 40 models of the superior dental arch prepared in dental stone. The arches were selected to be representative of different grades of alignment and to include different stages of mixed and permanent dentition. The points used to measure the arch were the vertices of mesovestibular cuspsids of molar teeth, vestibular cuspsids of premolar teeth, and canine cuspsids. The center of the incisal edge was also registered, mesodistally as well as buccolingually, in the central and the lateral teeth. All these reference points were digitized through the use of a bidimensional coordinate system. The precision was .02 mm (Figs 1a to 1c). The information was processed with the MATLAB program (Mathworks, Natick, MA).
In order to define the arch, an approximation was made through fourth-order polynoma of the form: 
\[ y = a + bx + Cx^2 + dx^3 + ex^4 \]
and then as a trigonometric series of periodic functions known as Fourier transforms. Fourier series are mathematical functions that describe an outline: Complex forms are divided into a series of cosine and sine functions of increasing frequency, which can be used to compare different outlines.\textsuperscript{13} The mathematical model of Fourier approximation is extensively described in the literature,\textsuperscript{14} and it has been successfully applied in neurology, dentistry,\textsuperscript{15} osteology, and hematology. Its application to the form of dental arch can be described in the following way:

**Periodic function**

An \( f(x) \) function is considered periodic if it is defined for any real value of \( x \) and if there is a positive integer \( T \) that makes:

\[ (1) \quad f(x + T) = f(x) \] for any value of \( x \)

\( N \) being any integer, then:

\[ (2) \quad f(x + nT) = f(x) \] for any value of \( x \)

So that \( 2T, 3T, \) and \( 4T \) are also periods of the function \( f(x) \). And \( T \) receives the name of period of \( f(x) \).

The graphic representation of this kind of function is obtained by the periodic repetition of the graph obtained for any interval of length \( T \).

**Fourier series**

Assuming that \( f(x) \) is a periodic function with a \( T \) period, that may be represented by the trigonometric series:

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{2n\pi}{T} x \right) + b_n \sin \left( \frac{2n\pi}{T} x \right) \right] \]

\( A_0, a_n, \) and \( b_n \) being the coefficients of the Fourier series, obtained by Euler formulae, for \( n = 1, 2, 3... \)

**Odd and even function**

\[ a_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos \left( \frac{2n\pi}{T} x \right) \, dx \]

\[ b_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \sin \left( \frac{2n\pi}{T} x \right) \, dx \]

\( \text{Figs 1a to 1c} \) Bidimensional coordinate system used to measure cuspids.
The function \( y = f(x) \) is considered even (or symmetric) if:

\[
(4) \quad f(-x) = f(x)
\]

The same function is considered odd if:

\[
(5) \quad f(-x) = -f(x)
\]

Figure 2 shows the corresponding graphic representations to each function. The graph for an even function is symmetrical to the Y axis, while the graph for an odd function is antisymmetric to the same axis. For the even function it is true that:

\[
(6)
\]

while for the odd function:

\[
(7)
\]

The Fourier series for an even function, \( f(x) \), is a Fourier sinusoidal series, of \( T \) period expressed as:

\[
(8)
\]

With coefficients:

\[
(9)
\]

Development of equations to define the dental arch by Fourier series.

\( P \) being the number of reference points (cusps or interdental papillae) used to define the form of the dental arch, then the Fourier expansion for \( P \) points is expressed by the equation (8) using:

\[
(10)
\]

\( T \) in the equation (10) is the maximum width of the dental arch at the level of the mesial papilla of the first molar (P16), or at the mesial cuspid of the first permanent molar (C16). The aim was to analytically determine the expression for \( b_n \) to be used in the previously shown equation.

The Fourier approximation to the arch is obtained segmenting the arch in \( P-1 \) fragments, corresponding to the intervals between points of reference (papilla to papilla or cuspid to cuspid), as is shown in Fig 3. When the function is segmented that way, the \( b_n \) coefficients can be expressed as:

\[
(11) \quad b_n = \frac{2}{T} \sum_{i=1}^{P-1} b_{n,i}
\]

Where the \( b_{n,i} \) coefficients for each interval are defined as:

\[
(12)
\]

Notice that \( n \) makes reference to the coefficient number while \( i \) refers to the interval number. In the equation (12) the following definitions are involved:

\[
(13) \quad k = \frac{n\pi}{T}
\]

\[
(14) \quad m_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}
\]

\[
(15) \quad c_i = y_i - m_ix_i
\]

\( m_i \) is the slope for the interval \([x_i, x_{i+1}]\) and \( c_i \) is the intercept point of the segment shown in Fig 3 with the y axis. The expression (12), integrated for the \( x \) variable, produces:

\[
(16) \quad b_{n,i} = \frac{-\cos(kx_{i+1})}{k} \frac{y_{i+1}}{k} + \frac{\cos(kx_i)}{k} \frac{y_i}{k} + \frac{m_i}{k^2} \left[ \sin(kx_{i+1}) - \sin(kx_i) \right]
\]

To simplify the notation, the term (17) can be introduced:

\[
(17) \quad h_{i+1} = x_{i+1} - x_i
\]

This expresses the \( x \) axis distance between dental cusps. By including this expression (17), in the equation (16), the result obtained is:

\[
(18) \quad b_{n,i} = \frac{y_{i+1}}{k^2 h_{i+1}} \left[ -kh_{i+1} \cos(kx_{i+1}) + \sin(kx_{i+1}) - \sin(kx_i) \right] + \frac{y_i}{k^2 h_{i+1}} \left[ kh_{i+1} \cos(kx_i) - \sin(kx_{i+1}) + \sin(kx_i) \right]
\]

and then, by substituting the previous relation in the \( b_n \) coefficients definition, is finally obtained the expression:
This analytically represents the Fourier coefficients, and physically contains the information necessary about anteroposterior depth of the arch.

Figure 4 shows the spectrum and graphic expression of the arch formed by Fourier series in mixed and permanent dentition. The spectrum relates frequency with amplitude of the sinusoidal wave. The frequency is the number of repetitions of the function per unit of distance (equation 10) and corresponds, in the arch, to the distance between homologous points. The amplitude is registered in the $y$ axis of the first graph and represents the depth (anteroposterior distance) of the wave.

Figure 4b shows in detail the spectrum for case 4319. The first frequency indicates the maximum transversal distance in the posterior segment of the arch that generally corresponds to the first permanent molars (C16 to C26) and gives the arch width in the posterior segment. By the same way, the first segment represents the maximum amplitude of the arch, given by the $b_n$ coefficients described in equation (19). The series delineated in red color is for mixed dentition and the black represents permanent dentition. The arch obtained by the Fourier series is presented in Fig 4c.

### Results

The reproducibility error of the method was calculated for the incisal edge of the right central incisor (B11), using duplicate determinations in the models and applying the Dahlberg formula:

$$ \sqrt{\frac{\sum_{i=1}^{n} \text{Dif}_i^2}{2n}} $$

In this formula, $\text{Dif}_i$ is the difference between 2 paired determinations and $n$ is the number of determinations. The reproducibility errors fall in a range between 0.16 and 0.24 mm.

To analyze the form from the measures taken for each point in the $x$ and $y$ axis, fourth-grade equations were obtained from the 40 models for the dental arch. The form of the quartic equations is $y = ax^4 + bx^3 + cx^2 + dx + e$. The line representing the polynomial tendency represents the form of the dental arch or the arch determined by papillae in each case. At the same time, Fourier series were developed to describe the form of the dental arch for each model. By that system, each case was represented by quartic equations and by Fourier transforms as well.
Figure 5 reveals the general arch generated for the model 346 in permanent dentition. The graph on the left refers to the arch defined by the quartic equation. The middle graph corresponds to the same case represented by Fourier transforms. In this patient, the tendency generated by the fourth-grade equation is moderately well adjusted to the points forming a parabola. The errors detected when either the quartic or Fourier equations results are revised show that the Fourier transforms give a better fit to the points. In this case, the Fourier method errors are in the range between 0.01 and 0.03 mm, while the quartic equation gave errors between 0.32 and 2.12 mm. The curve generated by the equation is not reflecting the real arch form, because the method polishes the form passing at significant distance of the most unfitting points.

To yield a quantitative estimation of the difference between both methods, the error for each of the 40 models was calculated. This error was expressed as the standard deviation of the differences between calculated and real value of the y coordinate, measured in the model. Table 1 shows the error results, the average difference between methods, its degree of significance, and the ratio between errors in both methods. The results indicate that the error is consistently higher for the arch obtained using quartic equations as compared to the error for the arch obtained by the Fourier series. It is also evident that the difference is significant and that the error using Fourier transforms is 10% or less than the error found when quartic equations are used.
Discussion

This study indicates that it is possible to define the dental arch more precisely using the Fourier series than fourth-grade equations. Quartic equations have been the most used method to express the form of the dental arch, but they present some limitations when the arch is moderately to severely irregular. The method generates curves that fit to the real points in a variable way. If the points are relatively well fitted, the curve passes over or very near to them. But if the curve is quite irregular, the difference in distance increases in correlation to the irregularity. Although, it is theoretically possible to solve the problem increasing the terms of the polynomium, in practical terms the coefficients added are so low that this approach does not improve reliability.

To obtain the Fourier series, it is necessary to assume that the arch form is a periodic function, thereby adjusting the posterior segment of the arch so that the point for the last molar (relative to the initial point of the curve) coincides with the y axis zero. These assumptions are consistent with the viability of the method from the biological viewpoint. Therefore, we suggest to change the use of fourth-grade equations for the Fourier approach here presented. Mathematically, this method is more complex, but the number of terms employed and the availability of computer systems makes it reasonably feasible.

Conclusions

Fourier series provide a method to describe the form and size of the dental arch more precisely than quartic equations. Although this method is more complex, the accuracy improved is worth the effort and the availability of computer programs to help in the calculation makes its application feasible.

Acknowledgments

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Table 1 Error comparisons between Fourier transforms and quartic equations

<table>
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<tr>
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<th>Fourier transforms</th>
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<th>Quartic equations</th>
<th>Differences F/E</th>
<th>Error F/Error Q</th>
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<td>SD</td>
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<td>SD</td>
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AD = average difference.
SD = standard deviation.
P ≤ .001.
References